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TECHNICAL NOTE 2471

UNSTEADY LAMINAR BOUNDARY-LAYER FLOW

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SUMMARY

The laminar compressible boundary layer over an insulated flat plate moving with a time-dependent velocity has been analyzed in detail. A group of parameters arise which, if large, provide that the classical "starting from rest" solution applies, and, if small, that the motion is quasi-steady. These parameters relate to the time required for temporal changes to diffuse through the boundary layer. Deviations from the quasi-steady velocity and temperature profiles have been computed.

Unsteady laminar flows with pressure gradient and, probably, unsteady turbulent boundary layers are governed by similar parameters, which may be estimated in order to provide a criterion as to whether quasi-steadiness may be assumed for a given problem. The case of a fluctuating velocity field passing over a flat plate is discussed from this point of view.

INTRODUCTION

The unsteady laminar boundary-layer flow over various bodies starting from rest has been analyzed by several investigators, the first of whom was Blasius (see paragraph 65, reference 1). These treatments have been concerned with such matters as the onset of separation on an airfoil and the transient effects of impulsive start; they employ an expansion of some suitable quantity in powers of the time elapsed since the start of motion. Thus, only the earliest phase of motion is considered.

Unsteady boundary-layer flow for longer times elapsed since starting and for higher speeds require consideration. For example, the flight speed of the usual rocket missile varies continuously over the entire trajectory, and thus the important boundary-layer effects of skin friction and heat transfer must, in principle, be regarded as unsteady for the entire flight. Recently, Kaye (reference 2) has discussed the problem of unsteady heat-transfer effects associated

with missile flight, and has tacitly assumed that at high speeds the boundary layer responds with no time lag to changes in stream velocity; that is, that the boundary layer at any instant is that which would be associated with steady motion at the stream conditions prevailing at the same instant. Such a boundary layer is hereinafter called quasi-steady.

Another type of problem of unsteady boundary-layer flow for high speed and long elapsed time concerns the response of a boundary layer to time-dependent velocity or pressure fluctuations in an otherwise steady outer potential flow. 2229

In the present report of research conducted at the NACA Lewis laboratory, the case of compressible laminar flow over a semi-infinite flat plate in rectilinear accelerated flight through still air is treated in detail; the flight speed is considered to vary with time in a continuous but otherwise arbitrary manner. The solution to this problem, which is intended as an idealization of missile flight, will be interpreted to provide insight into a more general class of unsteady boundary-layer flows, both laminar and turbulent.

The notation used in this report is described in appendix A.

ANALYSIS OF FLOW OVER FLAT PLATE IN UNSTEADY FLIGHT

Equations of Motion

The motion to be considered is that of a semi-infinite flat plate moving in a straight path, normal to its leading edge and in its own plane. Aside from the disturbance due to the motion of the plate, the medium (air) is at rest. The speed of flight may vary with time. A rectangular coordinate system (fig. 1(a)) is chosen to be at rest in the fluid, such that the plate moves with a velocity $U(t)$ in the negative x-direction, and y is measured normal to the plate. The origin of coordinates is taken to be the leading-edge location at $t = 0$.

The Prandtl boundary-layer assumptions are presumed to be valid for this flow and, in particular, the pressure is taken to be constant throughout the fluid. The Prandtl number and specific heats are assumed to be constant. Thus, the appropriate equations of motion of a compressible fluid are:

$Pr = \frac{C_p \mu}{\lambda}$

Omit C_p and Pr $\mu \sim T$

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$$\left.\begin{aligned} &\rho\left[\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y}\right] = \frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \\ &\rho\left[\frac{\partial (C_p\theta)}{\partial t} + u\frac{\partial}{\partial x}(C_p\theta) + v\frac{\partial}{\partial y}(C_p\theta)\right] = \frac{C_p}{Pr}\frac{\partial}{\partial y}\left(\mu\frac{\partial \theta}{\partial y}\right) + \cancel{\mu\left(\frac{\partial u}{\partial y}\right)^2} \\ &\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x}(\rho u) + \frac{\partial}{\partial y}(\rho v) = 0 \end{aligned}\right\} \quad (1)$$

$$\rho\theta = \text{constant}$$

These equations may be written in a new coordinate system (fig. 1(b)) fixed with reference to the plate, with the origin at the leading edge. Temporarily denoting by primes the new coordinates and the corresponding velocity components, the appropriate transformations are:

$$v' \equiv v; \quad u' \equiv u + U;$$

$$x' \equiv x + \int_0^t U\,dt; \quad y' \equiv y; \quad t' \equiv t$$

Equations (1) become:

$$\left.\begin{aligned} &\rho\left[\frac{\partial u'}{\partial t'} + u'\frac{\partial u'}{\partial x'} + v'\frac{\partial u'}{\partial y'}\right] = \cancel{\rho\frac{dU}{dt'}} + \frac{\partial}{dy'}\left(\mu\frac{\partial u'}{\partial y'}\right) \\ &\rho\left[\frac{\partial \theta}{\partial t'} + u'\frac{\partial \theta}{\partial x'} + v'\frac{\partial \theta}{\partial y'}\right] = \frac{1}{Pr}\frac{\partial}{\partial y'}\left(\mu\frac{\partial \theta}{\partial y'}\right) + \frac{\mu}{C_p}\left(\frac{\partial u'}{\partial y'}\right)^2 \\ &\frac{\partial \rho}{\partial t'} + \frac{\partial}{\partial x'}(\rho u') + \frac{\partial}{\partial y'}(\rho v') = 0 \end{aligned}\right\} \quad (2)$$

$$\rho\theta = \text{constant}$$

Hereinafter, the primes will not be used, and x, y, t, u, and v are understood to be measured relative to the coordinate system fixed in the plate (fig. 1(b)).

The temperature-viscosity relationship discussed by Chapman and Rubesin in reference 3 is employed; that is, it is assumed that the variation of viscosity with temperature may be approximated by the relation

$$\mu = \mu_\infty C \frac{\theta}{\theta_\infty} \equiv \rho_\infty \nu \frac{\theta}{\theta_\infty} \tag{3}$$

where C is a constant obtained by matching equation (3) to the Sutherland formula at (for example) the wall. This matching yields, according to reference 3,

$$C = \sqrt{\frac{\theta_w}{\theta_\infty}}\left(\frac{\theta_\infty + 216^\circ \text{ R}}{\theta_w + 216^\circ \text{ R}}\right) \tag{4}$$

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The equations of motion are simplified by the use of a transformation of coordinates, similar to that employed in reference 4, and of a stream function, as follows:

$$\left.\begin{aligned}
&Y \equiv \int_0^y (\rho/\rho_\infty)\,dy; \quad X \equiv x; \quad T \equiv t \\
&u \equiv \frac{\rho_\infty}{\rho}\frac{\partial\psi}{\partial y} = \frac{\partial\psi}{\partial Y} \\
&v \equiv -\frac{\rho_\infty}{\rho}\left(\frac{\partial\psi}{\partial x} + \frac{\partial}{\partial t}\int_0^y \frac{\rho}{\rho_\infty}\,dy\right) = -\frac{\rho_\infty}{\rho}\left(\frac{\partial\psi}{\partial X} + \frac{\partial Y}{\partial x}\frac{\partial\psi}{\partial Y} + \frac{\partial Y}{\partial t}\right)
\end{aligned}\right\} \tag{5}$$

Relations (3) and (5) transform the equations (2) of motion to yield:

$$\psi_{YT} + \psi_Y\psi_{XY} - \psi_X\psi_{YY} = U'(T) + \nu\psi_{YYY} \tag{6a}$$

$$\theta_T + \psi_Y\theta_X - \psi_X\theta_Y = \frac{1}{Pr}\nu\theta_{YY} + \frac{\nu}{c_p}(\psi_{YY})^2 \tag{6b}$$

The appropriate boundary conditions on ψ are

$$\psi_Y(X,\infty) = \psi_Y(0,Y) = U(T) \tag{7a}$$

$$\psi_Y(X,0) = 0 \tag{7b}$$

$$\psi(X,0) = 0 \tag{7c}$$

The boundary conditions on $\theta(X,Y)$ must be chosen with regard to the particular problem under consideration. (Subscripts and primes denote partial and ordinary differentiation, respectively.)

Solutions for ψ, in the Case of Constant Acceleration

If the velocity is given by the relation $U(T) = AT$, time T being measured from rest, the equations (6) may be written in similarity form, defining

$$\psi \equiv \sqrt{\nu}\, AT^{3/2}\, g(Z,\zeta)$$

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$$Z \equiv Y/\sqrt{\nu T}; \quad \zeta \equiv X/AT^2$$

These definitions enable equations (6a) and (7) to be written as follows:

$$g_Z - \frac{1}{2} Z g_{ZZ} - 2\zeta g_{Z\zeta} + g_Z g_{Z\zeta} - g_\zeta g_{ZZ} = 1 + g_{ZZZ} \tag{8}$$

$$g_Z(Z,0) = g_Z(\infty,\zeta) = 1; \quad g_Z(0,\zeta) = g(0,\zeta) = 0 \tag{9}$$

Equation (8) is obtained by considering $\nu = \frac{\mu_\infty}{\rho_\infty} C$ to be a constant. Subject to the limitations imposed by this assumption, which is discussed in appendix B, this analysis may be considered applicable to compressible flow.

Case $\zeta >> 1$. - If $\zeta = X/AT^2 >> 1$, the function $g(Z,\zeta)$ may be expanded in reciprocal powers of ζ as follows:

$$g = g_0(Z) + \frac{1}{\zeta} g_1(Z) + \frac{1}{\zeta^2} g_2(Z) + \ldots$$

$$g_0'(\infty) = 1; \quad g_1'(\infty) = g_2'(\infty) = \ldots = 0$$

$$g_0'(0) = 0; \quad g_1'(0) = g_2'(0) = \ldots = 0$$

$$g_0(0) = 0; \quad g_1(0) = g_2(0) = \ldots = 0$$

When, in equation (8), successive reciprocal powers of ζ are equated, this definition and set of boundary conditions yield a sequence of differential equations, the first of which is

$$g_0' - \frac{1}{2} Z g_0'' = 1 + g_0'''$$

and the solutions

$$g_0' = -\frac{1}{2}Z^2 + (\pi)^{-1/2}Ze^{-Z^2/4} + (Z^2 + 2)\ (\pi)^{-1/2}\int_0^{Z/2} e^{-z^2}\,dz$$

$$g_1 = g_2 = g_3 = \ldots = 0$$

This approach provides a solution equivalent to that for the initial motion of a flat plate with constant acceleration presented in paragraph 66, reference 1. 2229

This analysis clearly pertains to the very earliest stages of motion since the function $g(Z,\zeta)$ is expanded, in effect, in powers of time elapsed since the start of motion. In this connection, it may be noted that the inertia terms in the momentum equation drop out as a consequence of assuming $\zeta >> 1$.

The boundary condition $g_Z(Z,0) = 1$ has not been applied. Thus, no consideration is given to the growth of the boundary layer in the X-direction, and the effect of the leading edge is ignored. The quantity ζ is equivalent to the ratio of the distance (X) aft of the leading edge to the distance $\left(\frac{1}{2}AT^2\right)$ that the plate has traveled. (See fig. 1(a).) If $\zeta >> 1$, it is therefore proper to consider that the effect of the leading edge has not yet been felt at the station X.

Case $\zeta << 1$. - If, however, the distance traveled is much larger than distance aft of the leading edge, that is, if $\zeta << 1$, then account must be taken of the growth with X of the boundary layer. The appropriate expansion of $g(Z,\zeta)$ for small values of ζ is therefore

$$g = \sqrt{\zeta}\left[g_0\left(\frac{Z}{2\sqrt{\zeta}}\right) + \zeta g_1\left(\frac{Z}{2\sqrt{\zeta}}\right) + \zeta^2\ g_2\left(\frac{Z}{2\sqrt{\zeta}}\right) + \ldots\right]$$

subject to the boundary conditions

$$g_0'(\infty) = 2;\quad g_1'(\infty) = g_2'(\infty) = \ldots = 0$$

$$g_0'(0) = 0;\quad g_1'(0) = g_2'(0) = \ldots = 0$$

$$g_0(0) = 0;\quad g_1(0) = g_2(0) = \ldots = 0$$

Substitution of this expansion of $g(Z,\zeta)$ into equation (8) yields a sequence of differential equations the first of which is

$$g_0 g_0'' + g_0''' = 0$$

This differential equation and the corresponding set of boundary conditions are precisely those which apply in the case of steady flow over a semi-infinite flat plate. (See reference 5.) In the steady case, the argument of g_0 is $\frac{1}{2}\frac{Y}{\sqrt{X}}\sqrt{\frac{U}{\nu}}$. In the present analysis, the argument is

$$\frac{1}{2}\frac{Z}{\sqrt{\zeta}} = \frac{1}{2}\frac{Y}{\sqrt{\nu T}}\bigg/\sqrt{\frac{X}{AT^2}} = \frac{1}{2}\frac{Y}{\sqrt{X}}\sqrt{\frac{AT}{\nu}} = \frac{1}{2}\frac{Y}{\sqrt{X}}\sqrt{\frac{U(T)}{\nu}}$$

Therefore, at any instant when $\zeta << 1$, the boundary-layer flow for constant acceleration is nearly that which would obtain in steady flow at a velocity equal to the instantaneous velocity of the accelerated plate; as $\zeta \rightarrow 0$, the equivalence becomes precise. The correction terms arising when ζ is small but not zero will be considered in a subsequent section.

Parameters for Unsteady Flat-Plate Flow

Before the flow over a flat plate with an arbitrary velocity-time relation is considered, it is desirable to determine the governing parameters. From the important physical quantities X, ν, U, $U'(T)$, $U''(T)$, . . . $U^{(n)}(T)$, . . . the following dimensionless quantities may be constructed (in addition to Reynolds number and Mach number):

$$\frac{XU'}{U^2};\quad \frac{X^2U''}{U^3};\quad \ldots;\quad \frac{X^nU^{(n)}}{U^{n+1}};\quad \ldots \tag{10}$$

Quantities (10) and Reynolds number are therefore the quantities expected to govern the unsteady incompressible boundary layer on a flat plate with arbitrary (but differentiable) unsteady flight speed. These quantities, in the same form, also govern the corresponding compressible flow, as will be shown in the next section.

If $U = AT$, as in the previous section, then the quantities (10) specialize to the single parameter $\zeta = \frac{X}{AT^2}$. This quantity ζ has been noted to be equivalent to the ratio of the distance aft of the leading edge to the distance traveled by the plate.

A more direct physical meaning may be attached to the quantities (10), and hence to the special case ζ, as follows:

If some physical quantity were suddenly to change at the boundary of a fluid region of thickness δ in which conditions were initially uniform, this change would diffuse, with time, throughout the region; uniformity would ultimately be established again. The time for this diffusive process to reach some prescribed degree of completion will, of course, depend on the distance δ through which it must act and on the coefficient of diffusion ν of the fluid (kinematic viscosity for dynamical changes). In particular, one-dimensional diffusion theory indicates that the time required is proportional to δ^2/ν.

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Temporal changes at the edge of a boundary layer will be evened out, or diffused, through the layer by this mechanism. Accordingly, the time δ^2/ν will be taken as characteristic of the time required to diffuse throughout the layer a unit proportional change occurring at the outer edge. If the boundary layer is substantially the quasi-steady flow over a flat plate, δ^2 is approximately proportional to $\nu X/U$, and the "diffusion time" is proportional to X/U.

Owing to the unsteadiness of the flow, changes in U appear at the edge of the boundary layer. The time required for a unit proportional change in stream velocity to take place cannot be given by a single quantity, but, rather, will be characterized by the following set of times:

$$\frac{U}{U'}\ ;\ \left(\frac{U}{U''}\right)^{1/2};\ \left(\frac{U}{U'''}\right)^{1/3};\ \ldots$$

Accordingly, the ratio of "diffusion time" X/U to the time required for a unit proportional change to be imposed is characterized by the following group of quantities:

$$\frac{XU'}{U^2}\ ;\ X\left(\frac{U''}{U^3}\right)^{1/2};\ X\left(\frac{U'''}{U^4}\right)^{1/3},\ \ldots$$

These quantities are equivalent to the quantities (10), which may thus be considered a measure of the promptness with which the boundary layer responds to impressed changes. Clearly, then, if the quantities (10) are very small, quasi-steady flow is to be expected. If the velocity is increasing, the boundary layer becomes progressively thinner and hence responds more and more quickly to changes at its outer edge. This effect clearly tends to establish quasi-steady flow.

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Solution for Insulated Plate and

Arbitrary Velocity-Time Relation

Solutions of the compressible-flow equations (6) and (7) will be sought for the case of a flat insulated plate traveling with a speed which may vary with time in a differentiable though otherwise arbitrary way.

If consideration is restricted to a stage of motion where the diffusion time is already rather small, the stream function may be defined as

$$\psi \equiv U\sqrt{\frac{\nu X}{U}}\, f(\sigma, \zeta_0, \zeta_1, \zeta_2, \ldots) \tag{11}$$

where $\sigma \equiv \frac{1}{2}\frac{Y}{\sqrt{X}}\sqrt{\frac{U}{\nu}}$, and ζ_n is a function of X,T, the definition of which will be deduced in the course of the analysis. Similarly,

$$\frac{\theta}{\theta_\infty} \equiv 1 + \frac{\gamma-1}{2} M_\infty^{\ 2}(T)\, r(\sigma, \zeta_0, \zeta_1, \zeta_2, \ldots) \tag{12}$$

(The "recovery factor" is $r(0, \zeta_0, \zeta_1, \ldots)$.)

Substitution of definitions (11) and (12) into equations (6) and (7) yields

$$f_{\sigma\sigma\sigma} + ff_{\sigma\sigma} = -8\frac{XU'}{U^2} + 2\left(2\frac{XU'}{U^2} + X\sum_{n=0}^{\infty} f_{\sigma\zeta_n}\zeta_{nX}\right)f_\sigma - 2Xf_{\sigma\sigma}\sum_{n=0}^{\infty} f_{\zeta_n}\zeta_{nX} + 4\frac{X}{U}\sum_{n=0}^{\infty} f_{\sigma\zeta_n}\zeta_{nT} + 2\frac{XU'}{U^2}\sigma f_{\sigma\sigma} \tag{13}$$

$$r_{\sigma\sigma} + \Pr fr_\sigma + \frac{1}{2}\Pr(f_{\sigma\sigma})^2 = 4\Pr\left(2\frac{XU'}{U^2}r + \frac{1}{2}\frac{XU'}{U^2}\sigma r_\sigma + \frac{X}{U}\sum_{n=0}^{\infty} r_{\zeta_n}\zeta_{nT} + \frac{1}{2}Xf_\sigma\sum_{n=0}^{\infty} r_{\zeta_n}\zeta_{nX} - \frac{1}{2}Xr_\sigma\sum_{n=0}^{\infty} f_{\zeta_n}\zeta_{nX}\right) \tag{14}$$

$$f_\sigma(\infty, \zeta_n) = 2; \quad f_\sigma(0, \zeta_n) = f(0, \zeta_n) = 0 \tag{15}$$

Since the case of an insulated plate is to be treated, $\left(\frac{\partial \theta}{\partial Y}\right)_W = 0$; the function r thus satisfies the boundary conditions

$$r_\sigma(0,\zeta_n) = 0; \quad r(\infty,\zeta_n) = 0 \tag{16}$$

Equations (13) and (14) may be considered correct only if the quantity ν may be assumed constant. For constant wall temperature or for incompressible flow, equation (4) indicates that C, and hence ν, would be constant. However, under the condition of no heat transfer, the wall temperature, and hence ν, must depend to some degree on both X and T. The circumstances under which this dependence may be neglected are discussed in appendix B. 2229

Equations (13) and (14) may be made self-consistent (that is, containing only functions of σ and the various ζ_n) by defining

$$\zeta_0 \equiv \frac{XU'}{U^2}; \quad \zeta_1 \equiv \frac{X^2U''}{U^3}; \quad \zeta_2 \equiv \frac{X^3U'''}{U^4}; \ . \ . \ . \tag{17}$$

These definitions correspond to the quantities (10) previously derived by dimensional reasoning. If the quantities (17) are considered small, the functions f and r may be expanded as follows:

$$f(\sigma,\zeta_n) \equiv F(\sigma) + \zeta_0 f_0(\sigma) + \zeta_1 f_1(\sigma) + \ . \ . \ . + \zeta_0{}^2 f_{00}(\sigma) + \ . \ . \ . +$$
$$\zeta_0\zeta_1 f_{01}(\sigma) + \ . \ . \ . \tag{18}$$

$$r(\sigma,\zeta_n) \equiv R(\sigma) + \zeta_0 r_0(\sigma) + \zeta_1 r_1(\sigma) + \ . \ . \ . + \zeta_0{}^2 r_{00}(\sigma) + \ . \ . \ . +$$
$$\zeta_0\zeta_1 r_{01}(\sigma) + \ . \ . \ . \tag{19}$$

Substitution of equations (17), (18), and (19) into equations (13) through (16) and collecting terms multiplied by the various powers and products of the ζ_n yields, in part,

$$F''' + FF'' = 0 \tag{20a}$$

$$f_0''' + Ff_0'' - 2F'f_0' + 3F''f_0 = -4(2 - F') + 2\sigma F'' \tag{20b}$$

$$f_1''' + Ff_1'' - 4F'f_1' + 5F''f_1 = 4f_0' \tag{20c}$$

$$\vdots$$

$$F'(\infty) = 2; \quad F'(0) = F(0) = 0 \tag{21a}$$

$$f_n'(\infty) = f_n'(0) = f_n(0) = 0 \tag{21b}$$

$$R'' + \Pr FR' + \frac{1}{2}\Pr(F'')^2 = 0 \tag{22a}$$

$$r_0'' + \Pr Fr_0' - 2\Pr F'r_0 = \Pr\left(-F''f_0'' + 8R + 2\eta R' - 3R'f_0\right) \tag{22b}$$

$$r_1'' + \Pr Fr_1' - 4\Pr F'r_1 = \Pr\left(-F''f_1'' + 4r_0 - 5R'f_1\right) \tag{22c}$$

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$$R'(0) = R(\infty) = 0 \tag{23a}$$

$$r_n'(0) = r_n(\infty) = 0 \tag{23b}$$

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Equations (20a), (21a), (22a), and (23a) are the equations appropriate to steady flow. Accordingly, equations (18) and (19) indicate that for small values of the ζ_n the laminar compressible boundary layer on an insulated flat plate is nearly quasi-steady, with respect to both velocity and temperature profiles, provided that f_n and r_n and their derivatives are of unit order of magnitude.

Magnitudes of ζ_n. - The system of equations just derived will be valid only under circumstances in which the ζ_n are small, and (to be conservative) can be arranged in a decreasing sequence. The first, and usually dominant member of this group, is $\zeta_0 = \frac{XU'}{U^2}$. In practical situations, the requirement of small ζ_0 will commonly be met. For example, at a distance of 10 feet aft of the leading edge of a body having an instantaneous acceleration of 100 feet per second per second (about three times the acceleration due to gravity), the quantity ζ_0 will be less than 10 percent for a velocity greater than 100 feet per second and less than 1 percent for velocities greater than 316 feet per second. For the case of constant acceleration at 100 feet per second per second, ζ_0 will be less than 1 percent after the first 3.16 seconds of flight.

For constant acceleration, ζ_0 is the only nonzero ζ_n. For flight at a velocity U proportional to T^m,

$$\zeta_1 = \frac{(m-1)}{m}\zeta_0^2; \quad \zeta_2 = \frac{(m-1)(m-2)}{m^2}\zeta_0^3$$

and so forth. For this case, the ζ_n taken in order clearly form a decreasing sequence if ζ_0 itself is small and m is not too small. Since the successive ζ_n involve increasing powers of velocity in the denominator, it will be true that the ζ_n will generally form a diminishing sequence for moderate or high speeds, provided that $U(T)$ is a differentiable function. The higher the speed, the more sharply $U(T)$ may vary without violating this requirement.

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Solution of momentum equations. - The solution of equation (20a) subject to boundary conditions (21a) is available in reference 5. The following features of the function $F(\sigma)$ are important:

$$F''(0) = 1.328 \tag{24}$$

$$\lim_{\sigma\to\infty}\left[F(\sigma) - 2\sigma\right] = -1.721 \tag{25}$$

Equations (20b) and (20c) subject to boundary conditions (21b) have been solved by direct numerical integration. The method used is described in appendix C. Values of $f_0(\sigma)$, $f_1(\sigma)$, and their derivatives are presented in table I. The functions $F'(\sigma)$, $f_0'(\sigma)$, and $f_1'(\sigma)$ are plotted in figure 2.

The wall shear stress $\tau_w = \left(\mu \frac{\partial u}{\partial y}\right)_w$ may be obtained in the following dimensionless form:

$$\frac{\tau_w}{\frac{1}{2}\rho_\infty U^2} = \frac{1}{2}\sqrt{\frac{\nu}{UX}}\left[F''(0) + \frac{XU'}{U^2} f_0''(0) + \frac{X^2U''}{U^3} f_1''(0) + \ldots\right]$$

or, according to equation (24) and the results presented in table I,

$$\frac{\tau_w}{\frac{1}{2}\rho_\infty U^2} = (0.664)\sqrt{\frac{\nu}{UX}}\left[1 + (2.555)\frac{XU'}{U^2} - (1.414)\frac{X^2U''}{U^3} + \ldots\right] \tag{26}$$

The leading term of the right side is the quasi-steady solution. As would be expected, positive acceleration provides a skin friction higher than the quasi-steady value.

Displacement thickness. - The displacement thickness δ^* may be defined as follows:

$$\delta^* \equiv \int_0^\infty \left(1 - \frac{\rho}{\rho_\infty}\frac{u}{U}\right) dy$$

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or, by use of equations (5) and (11),

$$\delta^* = \sqrt{\frac{X\nu}{U}}\int_0^\infty \left[2\frac{\rho_\infty}{\rho} - f'(\sigma)\right] d\sigma = \sqrt{\frac{X\nu}{U}}\int_0^\infty \left[2\frac{\theta}{\theta_\infty} - f'(\sigma)\right] d\sigma$$

or, when equation (12) is introduced,

$$\delta^* = \sqrt{\frac{X\nu}{U}}\int_0^\infty \left[2 - f'(\sigma) + (\gamma-1)\, M_\infty^{\ 2}\, r\right] d\sigma$$

When equations (17) and (19) are used, the following equation results:

$$\delta^* = \sqrt{\frac{X\nu}{U}}\left\{\lim_{\sigma\to\infty}(2\sigma - F) - \left[\frac{XU'}{U^2}f_0(\infty) + \frac{X^2U''}{U^3}f_1(\infty) + \ldots\right] + (\gamma-1)\, M_\infty^{\ 2}\left[\int_0^\infty R\, d\sigma + \frac{XU'}{U^2}\int_0^\infty r_0 d\sigma + \frac{X^2U''}{U^3}\int_0^\infty r_1 d\sigma + \ldots\right]\right\}$$

When equation (25) and the information in tables I and II are applied for air having a Prandtl number of 0.72, this expression becomes

$$\delta^* = (1.721)\sqrt{\frac{X\nu}{U}}\left\{1 + (0.645)(\gamma-1)\, M_\infty^{\ 2} - (0.592)\frac{XU'}{U^2}\left[1 + (2.938)(\gamma-1)\, M_\infty^{\ 2}\right] - (0.694)\frac{X^2U''}{U^3}\left[1 - (2.702)(\gamma-1)\, M_\infty^{\ 2}\right] + \ldots\right\} \tag{27}$$

the leading term of which is the quasi-steady result. Positive acceleration results in a boundary layer which is thinner than the quasi-steady boundary layer.

Solution of energy equations. - The solution of equation (22a) is available (for example, in reference 3). The remaining equations (22b) and (22c), subject to boundary conditions (23b), have been solved for $Pr = 0.72$ by direct numerical integration. The method used is described in appendix C. As a check on numerical procedure, equation (22a) has been solved by the same method. Values of $R(\sigma)$, $r_0(\sigma)$, $r_1(\sigma)$, and

their first derivatives are presented in table II. The result $R(0) = 0.848$ agrees closely with the value 0.845 given in reference 3. The functions $R(\sigma)$, $r_0(\sigma)$, and $r_1(\sigma)$ are plotted in figure 3. The recovery factor for $Pr = 0.72$ is

$$r(0) = R(0) + \frac{XU'}{U^2} r_0(0) + \frac{X^2U''}{U^3} r_1(0) + \ldots$$

or, from table II,

$$r(0) = (0.848)\left[1 - (2.852)\frac{XU'}{U^2} + (3.266)\frac{X^2U''}{U^3} + \ldots\right] \tag{28}$$

the leading term of which is the quasi-steady solution (see reference 3). Positive acceleration results in a surface temperature lower than the quasi-steady value.

UNSTEADY FLOW IN PRESSURE GRADIENT

The foregoing analysis has shown that the boundary layer on a flat plate traveling with a speed which varies with time becomes quasi-steady for vanishing values of the parameters XU'/U^2, X^2U''/U^3, and so forth. When these quantities are small but not zero, proportional correction terms arise.

When the behavior of a boundary layer in unsteady flow with pressure gradient is to be investigated, it would appear expedient first to inquire whether the boundary layer may be considered quasi-steady. If this is not done, needless effort may be expended on an essentially simple problem. In order to establish a criterion for quasi-steady flow in a pressure gradient, knowledge is required of the significant parameters arising in the unsteady equations of motion.

Integrated Momentum Equation

The incompressible laminar boundary-layer equations for unsteady flow,

$$u_t + uu_x + vu_y = U_t + UU_x + \nu u_{yy} \tag{29}$$

$$u_x + v_y = 0 \tag{30}$$

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may be integrated to yield

$$\frac{\partial}{\partial t}\,(U\delta^*) + \frac{\partial}{\partial x}\,(U^2\delta^{**}) + UU_x\,\delta^* = \nu\left(\frac{\partial u}{\partial y}\right)_w \tag{31}$$

where x and y are measured along and normal to the wall, respectively (fig. 1(c)), and where

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$$\delta^* \equiv \int_0^\delta \left(1 - \frac{u}{U}\right) dy; \quad \delta^{**} \equiv \int_0^\delta \frac{u}{U}\left(1 - \frac{u}{U}\right) dy \tag{32}$$

where δ is considered a measure of the thickness of the boundary layer, as in the Pohlhausen analysis (see paragraph 60, reference 1). The velocity profile is written

$$\frac{u}{U} \equiv h(x,\eta); \quad \eta \equiv \frac{y}{\delta} \tag{33}$$

The differential equation (29) may be evaluated at $y = 0$ to yield

$$U_t + UU_x = -\nu(u_{yy})_w$$

whence, by use of definition (33),

$$-h_{\eta\eta}(x,0) = \frac{\delta^2 U_x}{\nu} + \frac{\delta^2 U_t}{\nu U}$$

or

$$h_{\eta\eta}(x,0) = -\Gamma \equiv -(\Lambda + \xi) \tag{34}$$

where

$$\Lambda = \frac{\delta^2 U_x}{\nu}$$ ("Pohlhausen Λ", see paragraph 60, reference 1)

$$\xi = \frac{\delta^2 U_t}{\nu U}$$

Equation (34) indicates that the functional form of h may be written

$$\frac{u}{U} = h(\Gamma,\eta) \tag{35}$$

whence,

$$\delta^* = \delta H^*(\Gamma); \quad \delta^{**} = \delta H^{**}(\Gamma) \tag{36}$$

Substituting equations (35) and (36) into equation (31) yields

$$\left(\frac{1}{2} H^* + \Gamma \frac{dH^*}{d\Gamma}\right) \frac{\partial}{\partial t}\left(\frac{\delta^2}{\nu}\right) + \left(\frac{\delta^4 U_{xt}}{\nu^2} + \frac{\delta^4 U_{tt}}{\nu^2 U} - \xi^2\right) \frac{dH^*}{d\Gamma} +$$

$$\Gamma H^* + \frac{\delta}{\nu U} \frac{\partial}{\partial x} \left(U^2 \delta H^{**}\right) = h_\eta(\Gamma, 0) \tag{37}$$

Equation (37) shows that, irrespective of the time variation of U and δ, the boundary layer will be quasi-steady if the following quantities are each sufficiently small:

$$\frac{\delta^2 U_t}{\nu U}; \quad \frac{\partial}{\partial t}\left(\frac{\delta^2}{\nu}\right); \quad \frac{\delta^4 U_{xt}}{\nu^2}; \quad \frac{\delta^4 U_{tt}}{\nu^2 U} \tag{38}$$

As has been mentioned previously, the quantity δ^2/ν may be regarded as a measure of the time required to diffuse a change through the boundary layer. Changes in time of either U or δ may give rise to effects to be diffused. Thus, the first and last of the quantities (38) may be regarded as the ratios of diffusion times to the times for unit changes of stream velocity to occur at a given value of x because of acceleration and rate of increase of acceleration, respectively. The second member of the group is proportional to $\frac{\delta^2}{\nu} \frac{1}{\delta} \frac{\partial \delta}{\partial t}$ and thus relates diffusion time to time for the occurrence of a unit change in δ.

Relative to a mass of fluid passing through the boundary layer, temporal changes may occur by reason of variation of U or δ with x. Measured in this way, the time for a unit change in velocity to occur is $1/U_x$. Forming the usual ratio with diffusion time yields $\frac{\delta^2}{\nu} U_x$, which is the "Pohlhausen Λ". If this parameter vanishes, the boundary layer is that appropriate to the case of zero pressure gradient. It is interesting to note the analogy with unsteady motion: "Quasi-flat-plate" flow will occur even if U_x does not vanish, provided that $\frac{\delta^2}{\nu} U_x$ is small enough; the requirement is that changes in U diffuse

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to the wall much more rapidly than they occur in the stream. Accordingly, the third member of the group (38) is analogous to the fourth but depends on both Eulerian and Lagrangian time variations of U.

It may further be noted that the quantity Γ (equation (34)) is related to the diffusion through the boundary layer of changes in shear stress due to stream velocity variation: Roughly, the shear is proportional to U/δ, and because of velocity variation alone, its total time derivative is $U_t/\delta + UU_x/\delta$. The ratio of diffusion time δ^2/ν to the time for unit proportional change of shear to occur is thus

$$\frac{\delta^2}{\nu}\frac{\delta}{U}\left(\frac{U_t}{\delta} + \frac{UU_x}{\delta}\right) = \frac{\delta^2 U_t}{\nu U} + \frac{\delta^2 U_x}{\nu} = \Gamma$$

For the case of zero pressure gradient, the parameters $\frac{\delta^2 U_t}{\nu U}$ and $\frac{\partial}{\partial t}\left(\frac{\delta^2}{\nu}\right)$ both reduce to the previously discussed quantity $\frac{xU'}{U^2}$, and the parameter $\frac{\delta^4 U_{tt}}{\nu^2 U}$ reduces to $\frac{x^2 U''}{U^3}$. Therefore, it is to be expected that $\frac{\delta^2 U_t}{\nu U}$ and $\frac{\partial}{\partial t}\left(\frac{\delta^2}{\nu}\right)$ will usually be of the same order and, for moderate or high speeds, will be much larger than either $\frac{\delta^4 U_{xt}}{\nu^2}$ or $\frac{\delta^4 U_{tt}}{\nu^2 U}$.

The foregoing discussion pertains directly only to incompressible flow although similar considerations might be expected to apply in compressible flow.

For small, but nonzero, values of xU'/U^2 and x^2U''/U^3, proportional correction terms of the same order of magnitude arise in formulas (26) to (28) for shear stress, displacement thickness, and recovery factor in flow with zero pressure gradient. Since xU'/U^2 and x^2U''/U^3 are special cases of the quantities (38), it is expected that for cases involving pressure gradient, the magnitudes of the quantities (38) indicate quantitatively the order of magnitude of departure from the quasi-steady condition.

In a specific problem, the quantities (38) may be estimated as follows: Since a criterion for quasi-steadiness is desired, the boundary-layer thickness δ may be estimated on the basis of quasi-steady flow. If the resulting values of quantities (38) are of such a magnitude as to indicate nearly quasi-steady flow, the assumption made

in estimating δ is proper. In many cases, this estimate of δ, and particularly of $\partial\delta/\partial t$, which appears in the second member of the group (38), can be made only by finding the complete quasi-steady solution, a gamble justified in view of the great difficulty in finding a complete unsteady solution free of such assumptions.

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Oscillating Stream Velocity

As an example to which the foregoing discussion applies, the stream velocity over a flat plate is represented as

$$U = U_1 + u_1 \sin 2\pi k\left(\frac{x}{U_1} - t\right) \tag{39}$$

which denotes a flow wherein a sinusoidal velocity disturbance of amplitude u_1 and frequency k is carried downstream with the mean velocity U_1; the maximum magnitudes of the quantities (38) are, from equation (39),

$$\left|\frac{\delta^2 U_t}{\nu U}\right|_{max} \approx 2\pi \, Re_\delta \frac{\delta k}{U_1}\frac{u_1}{U_1}\ ; \quad Re_\delta \equiv \frac{U_1\delta}{\nu}$$

$$\left|\frac{\delta^4 U_{xt}}{\nu^2}\right|_{max} \approx \left|\frac{\delta^4 U_{tt}}{\nu^2 U}\right|_{max} \approx \left(2\pi \, Re_\delta \frac{\delta k}{U_1}\right)^2 \frac{u_1}{U_1}$$

The quantity $\frac{\partial}{\partial t}\left(\frac{\delta^2}{\nu}\right)$ cannot be estimated properly in advance. Tentatively, it may be assumed that it is of the same order as $\delta^2 U_t/\nu U$, and then, if it is concluded that the magnitudes of the remaining quantities indicate quasi-steadiness, the quasi-steady solution may be found and then analyzed to provide, as a check, a more precise estimate of $\frac{\partial}{\partial t}\left(\frac{\delta^2}{\nu}\right)$.

Thus, if $Re_\delta \frac{\delta k}{U_1}\frac{u_1}{U_1} \ll 1$ and $\left(Re_\delta \frac{\delta k}{U_1}\right)^2 \frac{u_1}{U_1} \ll 1$, the flow in question may be regarded as quasi-steady. If, as a rough approximation, the formula for boundary-layer thickness for steady uniform laminar flow over a flat plate is used, and if it is assumed that $k = 10$ cycles per second, $U_1 = 100$ feet per second, and $u_1/U_1 = 0.1$, then for approximately the first foot downstream of the leading edge,

$$Re_\delta \frac{\delta k}{U_1}\frac{u_1}{U_1} \leq 0.01$$

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and

$$\left(\mathrm{Re}_{\delta}\,\frac{\delta k}{U_1}\right)^2 \frac{u_1}{U_1} \leq 0.001$$

UNSTEADY TURBULENT BOUNDARY LAYER

The physical interpretation of the governing parameters in laminar flow would seem to have sufficient generality to permit extension to the case of the turbulent boundary layer. The most important quantities would be

$$\frac{\delta^2 U_x}{\nu}\ ;\ \ \frac{\delta^2 U_t}{\nu U}\ ;\ \ \frac{\delta^2}{\nu}\frac{1}{\delta}\delta_t \qquad (40)$$

where ν is to be interpreted as an average diffusion coefficient. The coefficient of diffusion by turbulent mixing may be written lu', where l is a "mixing length" and u' is related to the intensity of turbulent fluctuations. If the assumption is made that $l \propto \delta$ and $u' \propto U$, then the parameters (40) may be written

$$\delta\,\frac{U_x}{U}\ ;\ \ \frac{\delta U_t}{U^2}\ ;\ \ \frac{\delta_t}{U}$$

In reference 6 it was shown that the parameter $\delta\,\frac{U_x}{U}$ may not be considered to govern the response of a turbulent boundary layer to an adverse pressure gradient. The evidence is inconclusive for favorable pressure gradient. Thus, the assumption made as to the form of the turbulent diffusion coefficient is of questionable value and only qualitative significance should be attached to the unsteady flow parameters $\delta U_t/U^2$ and δ_t/U. In the absence of pressure gradient, δ may be taken to be (very roughly) proportional to x; thus

$$\frac{\delta U_t}{U^2} \propto \frac{xU_t}{U^2}\ ;\ \ \frac{\delta_t}{U} \approx 0$$

Therefore, it would seem that for zero pressure gradient, if $\frac{xU_t}{U^2}$ is small enough, a boundary layer will tend to be quasi-steady whether it is laminar or turbulent.

Actually, for zero pressure gradient, the quantity xU_t/U^2 may be obtained without assuming a specific form for the coefficient of turbulent diffusion. The boundary-layer thickness δ may be thought

of as the result of lateral diffusion of vorticity from a point source (the leading edge of the plate) moving with a velocity U. (Compare paragraph 17, reference 1.) If this process occurs without the complicating effects of pressure gradient, one-dimensional diffusion theory would indicate that $\delta^2 = \nu\frac{x}{U}$, where ν is either a laminar (molecular) or turbulent diffusion coefficient, and x/U is the time elapsed since the source of vorticity passed the station x. Thus, the "diffusion time" δ^2/ν appearing in the quantities (39) would be given roughly by x/U, and the quantity $\frac{\delta^2 U_t}{\nu U}$ reduces directly to xU_t/U^2.

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Of course, the parameter xU_t/U^2 yields a rather unsatisfactory criterion for quasi-steadiness in the turbulent case because no indication has been given of the magnitude of departure from quasi-steadiness associated with small nonzero values of this parameter.

CONCLUSIONS

The laminar compressible boundary-layer flow over a flat plate moving with a time-dependent velocity has been analyzed in detail. In the course of this analysis, a group of parameters arise, the magnitudes of which determine the nature of the flow unsteadiness. If the parameters are very large, the classical "starting-from-rest" solution applies; if very small, the flow may be regarded as nearly quasi-steady; that is, at any instant the motion is nearly that which would be obtained in steady flow at the conditions prevailing at that instant. The deviations of velocity and temperature profiles (for an insulated plate) from the quasi-steady state have been computed. Relative to quasi-steady flow, constant acceleration results in a thinner boundary layer with greater skin friction and lower wall temperature.

These parameters governing unsteady flat-plate flow may be shown to be special cases of a group of quantities which compare the time required to diffuse a unit proportional change in some stream quantity to the wall and the time for a unit proportional change in that quantity to appear in the outer flow. For moderate or high speeds and rather large accelerations, the first member of this group is commonly very small. The generalized unsteady flow parameters appear explicitly in the integrated momentum equation for boundary-layer flow with an unsteady pressure gradient.

In various problems of unsteady boundary-layer motion, for example, in the presence of oscillations in stream velocity or pressure, these parameters provide an advance criterion as to whether the flow need be considered essentially unsteady or whether the motion may be considered quasi-steady.

In the absence of a pressure gradient, the unsteady turbulent boundary layer is expected to be governed by the same unsteady flow parameter as governs the corresponding laminar boundary layer.

Lewis Flight Propulsion Laboratory,
National Advisory Committee for Aeronautics,
Cleveland, Ohio, May 25, 1951.

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APPENDIX A - NOTATION

The following notation is used in this report

A	acceleration
C	constant appearing in temperature-viscosity relation
c_p	specific heat at constant pressure
F	related to stream function for flat plate in steady flow
f,g	functions (identified by numerical subscript) related to stream function for unsteady flat plate flow
H^*, H^{**}	dimensionless forms of δ^* and δ^{**}
h	dimensionless velocity profile
M	Mach number
Pr	Prandtl number
R	function related to temperature profile for steady flat-plate flow
r	functions (identified by numerical subscript) related to temperature profile for unsteady flat-plate flow
T, t	time
U	stream velocity in x-direction
u	velocity in X, x-direction
v	velocity in Y, y-direction
X, x	coordinate along wall
Y, y	coordinate normal to wall
Z	dimensionless coordinate $(Y/\Lambda\sqrt{\nu T})$
Γ	dimensionless parameter $(\Lambda + \xi)$
γ	ratio of specific heats
δ	boundary-layer thickness

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δ^*, δ^{**} displacement and momentum thicknesses

ζ dimensionless coordinate (x/AT^2)

η dimensionless coordinate (y/δ)

θ temperature

Λ Pohlhausen parameter $\left(\frac{\delta^2 U_x}{\nu}\right)$

μ coefficient of viscosity

ν constant $\left(C \frac{\mu_\infty}{\rho_\infty}\right)$

ξ dimensionless parameter $\left(\frac{\delta^2 U_t}{\nu U}\right)$

ρ density

σ dimensionless coordinate $\left(\frac{1}{2} \frac{Y}{\sqrt{X}} \sqrt{\frac{U}{\nu}}\right)$

ψ stream function

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Subscripts:

w evaluation at wall $(y = 0)$

∞ evaluation in stream $(y \rightarrow \infty)$

Subscript notation for partial differentiation is used when convenient. Primes are used to denote ordinary differentiation.

APPENDIX B

ASSUMPTION OF CONSTANT ν

Equations (20) and (22) are derived for constant ν. Derivatives of ν with respect to time appear in the following terms of the differential equation (6) when definitions (11) and (12) are introduced:

$$\left.\begin{aligned} \psi_{YT} &= \frac{1}{2} U' f_\sigma + \frac{1}{2} U \left[\left(\frac{U'}{U} - \frac{\nu_T}{\nu}\right)\frac{1}{2}\sigma f_{\sigma\sigma} + \sum_0^\infty f_{\sigma\zeta_n} \zeta_{nT}\right] \\ \frac{\theta_T}{\theta_\infty} &= \frac{\gamma-1}{2} r \frac{dM_\infty^2}{dt} + \frac{\gamma-1}{2} M_\infty^2 \left[\left(\frac{U'}{U} - \frac{\nu_T}{\nu}\right)\frac{1}{2}\sigma r_\sigma + \sum_0^\infty r_{\sigma\zeta_n} \zeta_{nT}\right] \end{aligned}\right\} \tag{B1}$$

The quantity $\nu_T/\nu = C_T/C$ may be obtained from equation (4). Since the flat plate is considered to be insulated and since this discussion is concerned only with orders of magnitude, θ_w may be taken equal to the stream stagnation temperature $\theta_\infty\left(1 + \frac{\gamma-1}{2} M_\infty^2\right)$. When it is noted that γ for air is about 7/5,

$$\frac{\nu_T}{\nu} = \frac{C_T}{C} = -\frac{1}{5}\frac{U'}{U}\frac{M_\infty^2}{1 + \frac{1}{5}M_\infty^2}\left(\frac{1 + \frac{1}{5}M_\infty^2 - \frac{216^\circ\ R}{\theta_\infty}}{1 + \frac{1}{5}M_\infty^2 + \frac{216^\circ\ R}{\theta_\infty}}\right)$$

Thus, ν_T may be neglected in expressions (B1) if

$$\left|\frac{U}{U'}\frac{C_T}{C}\right| = \frac{\frac{1}{5}M_\infty^2}{1 + \frac{1}{5}M_\infty^2}\left|\frac{1 + \frac{1}{5}M_\infty^2 - \frac{216^\circ\ R}{\theta_\infty}}{1 + \frac{1}{5}M_\infty^2 + \frac{216^\circ\ R}{\theta_\infty}}\right| \ll 1 \tag{B2}$$

If $M_\infty = 0$, $\left|\frac{U}{U'}\frac{C_T}{C}\right| = 0$; and if $M_\infty = \infty$, $\left|\frac{U}{U'}\frac{C_T}{C}\right| = 1$. If θ_∞ is about 430° R, $\left|\frac{U}{U'}\frac{C_T}{C}\right| = 0.07$ for $M_\infty = 1$, and 0.25 when $M_\infty = 2$. The errors in determination of f_0 and r_0 due to neglect of ν_T are of order $\left|\frac{U}{U'}\frac{C_T}{T}\right|$. Therefore, for reasonable values of θ_∞ and for subsonic Mach numbers, the dependence of ν on T may properly be neglected. Of course, for high speeds when ν_T might be important,

the correction to the quasi-steady solution tends to be of slight importance since the magnitude of the correction is governed by such quantities as XU'/U^2, which tends to be small for large velocities.

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Derivatives of ν with respect to X appear in various terms of equations (6) when definitions (11) and (12) are introduced. The quantity ν_X appears in all cases in the factor $1 + X \frac{\nu_X}{\nu}$ and may be neglected if $X \frac{\nu_X}{\nu} = X \frac{C_X}{C} << 1$. Equation (4) combined with equations (12) and (19) yields, for small ζ_0,

$$X \frac{C_X}{C} = \frac{1}{2} \frac{UC_T}{U'C}\left(\frac{XU'}{U^2}\right) r_0(0) + . . .$$

If $r_0(0)$ is taken to be of unit order, it may be shown that neglecting ν_X also involves an error in f_0 and r_0 of order $\left|\frac{U}{U'} \frac{C_T}{C}\right|$.

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APPENDIX C

INTEGRATIONS

Integration of Momentum Equations

The ordinary differential equation (20b) of third order for $f_0(\sigma)$ and the boundary conditions (21b) constitute a two-point boundary-value problem, which may be solved numerically as follows: The scale of σ is divided into intervals of 0.1. If the value of $f_0(\sigma)$ and its derivatives is known at five successive points, a fifth-degree polynomial may be matched to the five known and a sixth unknown value of $f_0'''(\sigma)$. This polynomial may be integrated to yield $f_0''(\sigma)$, $f_0'(\sigma)$, and $f_0(\sigma)$ at the sixth point in terms of the six values of $f_0'''(\sigma)$. The condition that $f_0(\sigma)$ and its derivatives must satisfy the differential equation (20b) at the sixth point serves to determine at that point the value of $f_0'''(\sigma)$, and hence the values of $f_0(\sigma)$, $f_0'(\sigma)$, and $f_0''(\sigma)$. Thus, given the solution for five consecutive points, the solution may be extended to the next point.

The foregoing numerical procedure may be used for single-point boundary-value problems, provided $f(\sigma)$ is completely known for the first five intervals. In this case, the first five values are found by expanding $f_0(\sigma)$ and $F(\sigma)$ in a Taylor's series about $\sigma = 0$.

The difficulty presented by the fact that boundary conditions are given at two points $(\sigma = 0, \infty)$ is overcome by splitting the original two-point problem into two single-point problems, as follows: The solution $f_0(\sigma)$ is written

$$f_0(\sigma) \equiv Cf_0^{(1)}(\sigma) + f_0^{(2)}(\sigma) \tag{C1}$$

where C is a constant. The quantity $f_0^{(1)}(\sigma)$ is taken to be the solution of the single-point boundary-value problem

$$\left.\begin{aligned} f_0^{(1)'''} + Ff_0^{(1)''} - 2F'f_0^{(1)'} + 3F''f_0^{(1)} = 0 \\ f_0^{(1)''}(0) = 1;\quad f_0^{(1)'}(0) = f_0^{(1)}(0) = 0 \end{aligned}\right\} \tag{C2}$$

and $f_0^{(2)}(\sigma)$, the solution of

$$f_0^{(2)'''} + Ff_0^{(2)''} - 2F'f_0^{(2)'} + 3F''f_0^{(2)} = -4(2 - F') + 2\sigma F''$$

$$f_0^{(2)''}(0) = f_0^{(2)'}(0) = f_0^{(2)}(0) = 0 \qquad \text{(C3)}$$

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Definitions (C1) to (C3) provide that $f_0(\sigma) = Cf_0^{(1)}(\sigma) + f_0^{(2)}(\sigma)$ will satisfy differential equation (20b) and will yield, at $\sigma = 0$,

$$f_0''(0) = C; \quad f_0'(0) = f_0(0) = 0$$

If, after $f_0^{(1)}$ and $f_0^{(2)}$ have been determined, C is taken equal to $-\lim_{\sigma \to \infty} \dfrac{f_0^{(2)'}(\sigma)}{f_0^{(1)'}(\sigma)}$, the boundary condition $f_0'(\infty) = 0$ will be satisfied, and the solution for $f_0(\sigma)$ may be written

$$f_0(\sigma) = -\left[\lim_{\sigma \to \infty} \frac{f_0^{(2)'}(\sigma)}{f_0^{(1)'}(\sigma)}\right] f_0^{(1)}(\sigma) + f_0^{(2)}(\sigma)$$

The number of intervals for which it is necessary to compute $f_0^{(1)}$ and $f_0^{(2)}$ is primarily determined by the accuracy desired for the constant C.

The solution of equation (20c) is also obtained by the foregoing method.

Integration of Energy Equations

Equations (22a) to (22c) are solved by the same method as is used for the momentum equations, except that a second-degree (rather than fifth-degree) polynomial is matched to three successive values of $r''(\sigma)$. This simplification is considered warranted by the fact that the energy equations are of lower order than the momentum equations.

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TABLE I - FUNCTIONS ASSOCIATED WITH VELOCITY PROFILE

σ	f_0	f_0'	f_0''	f_1	f_1'	f_1''
0	0	0	3.394120	0	0	-1.878370
.2	.057747	.529447	1.953432	-.036758	-.359945	-1.653824
.4	.194689	.802508	.829535	-.138740	-.642866	-1.147047
.6	.365840	.881837	.013897	-.286337	-.813374	-.557151
.8	.538543	.827366	-.512697	-.456399	-.869151	-.018114
1.0	.691664	.693876	-.783065	-.627640	-.828805	.396729
1.2	.814136	.528061	-.845153	-.783550	-.721068	.653481
1.4	.903266	.365638	-.760117	-.913840	-.577909	.753609
1.6	.962232	.229349	-.594847	-1.014405	-.428269	.724592
1.8	.995747	.128992	-.409600	-1.086252	-.293752	.610518
2.0	1.014474	.063862	-.247975	-1.133778	-.186499	.461844
2.2	1.023130	.026808	-.130397	-1.162855	-.109292	.313684
2.4	1.026420	.008672	-.057777	-1.179289	-.059187	.193292
2.6	1.027286	.001372	-.019894	-1.187871	-.029603	.108468
2.8	1.019088	-.000727	-.003696	-1.192011	-.013669	.055563
3.0	1.018913	-.000829	.001437	-1.193852	-.005818	.026086
3.2	1.018786	-.000433	.002081	-1.194603	-.002261	.011299
3.4	1.018711	-.000377	.001358	-1.194883	-.000770	.004460
3.6	1.018660	-.000175	.000668	-1.195151	-.000210	.001640
3.8	1.018637	-.000082	.000276	-1.195168	0	.000600
4.0	1.018622	-.000043	.000088	-1.195160	.000090	.000250
4.2	1.018617	-.000034	.000014	-1.195130	.000120	.000160
4.4	1.018612	-.000031	-.000012	-1.195100	.000140	.000140

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TABLE II - FUNCTIONS ASSOCIATED WITH TEMPERATURE PROFILE

σ	R	R'	r_0	r_0'	r_1	r_1'
0	0.848000	0	-2.418820	0	2.769860	0
.2	.835307	-.126789	-2.382630	.374156	2.678376	-.846230
.4	.797491	-.250398	-2.267210	.779209	2.455221	-1.335023
.6	.735902	-.362717	-2.073428	1.147468	2.160495	-1.573702
.8	.653919	-.452272	-1.814377	1.423885	1.837448	-1.634264
1.0	.557309	-.507224	-1.512653	1.569377	1.513939	-1.587060
1.2	.453863	-.519942	-1.196740	1.567789	1.207085	-1.472302
1.4	.352151	-.490678	-.894813	1.431645	.928167	-1.309995
1.6	.259817	-.428239	-.630473	1.200147	.685411	-1.113120
1.8	.182067	-.347426	-.417568	.926319	.484160	-.897970
2.0	.121126	-.262489	-.259398	.660089	.326110	-.684630
2.2	.076520	-.185622	-.150731	.435205	.208880	-.492490
2.4	.045900	-.123264	-.081592	.265658	.126810	-.334450
2.6	.026133	-.077049	-.040868	.149928	.072500	-.214940
2.8	.014108	-.045402	-.018722	.077734	.038410	-.131330
3.0	.007206	-.025243	-.007696	.036708	.019510	-.061160
3.2	.003468	-.013248	-.002726	.015507	.010000	-.034920
3.4	.001555	-.006565	-.000742	.005627	.004880	-.017880
3.6	.000631	-.003073	-.000088	.001555	.002340	-.008470
3.8	.000209	-.001358	.000053	.000132	.001200	-.003640
4.0	.000027	-.000567	.000032	-.000242	.000720	-.001320
	$\int_0^\infty R\,d\sigma = 1.10974$		$\int_0^\infty r_0\,d\sigma = -2.99236$		$\int_0^\infty r_1\,d\sigma = 3.22928$	

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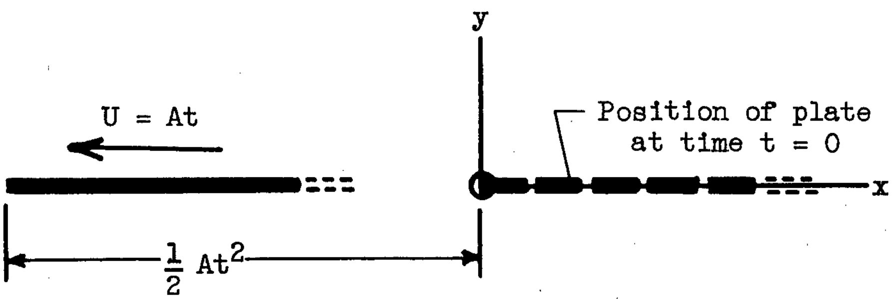
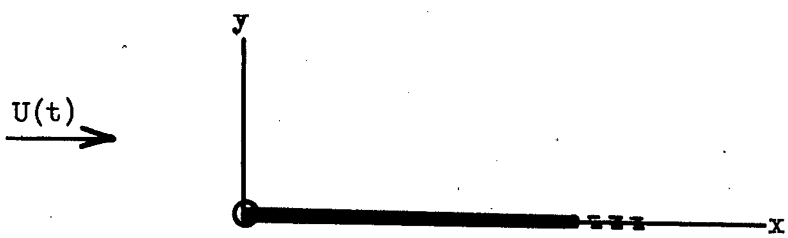
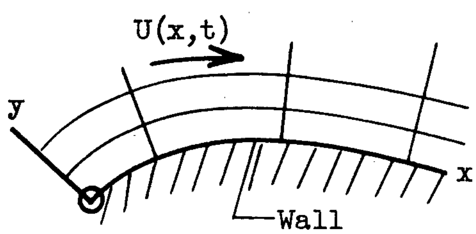


(a) Coordinates fixed in fluid at rest.



(b) Coordinates fixed in plate.



(c) Curvilinear coordinates; y = 0 on wall.

Figure 1. - Coordinate systems used in analysis.

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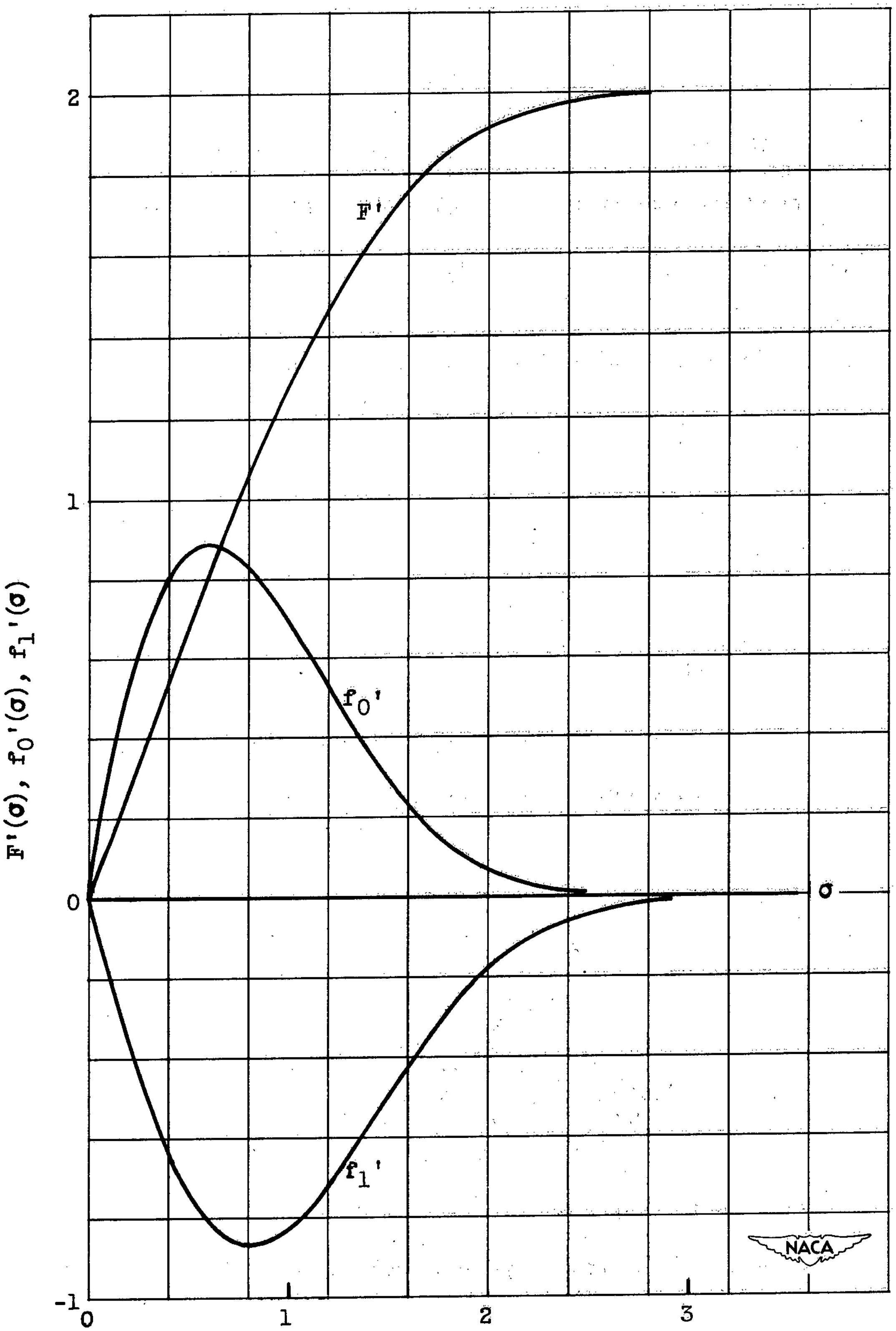

Figure 2. - Functions associated with velocity profile.

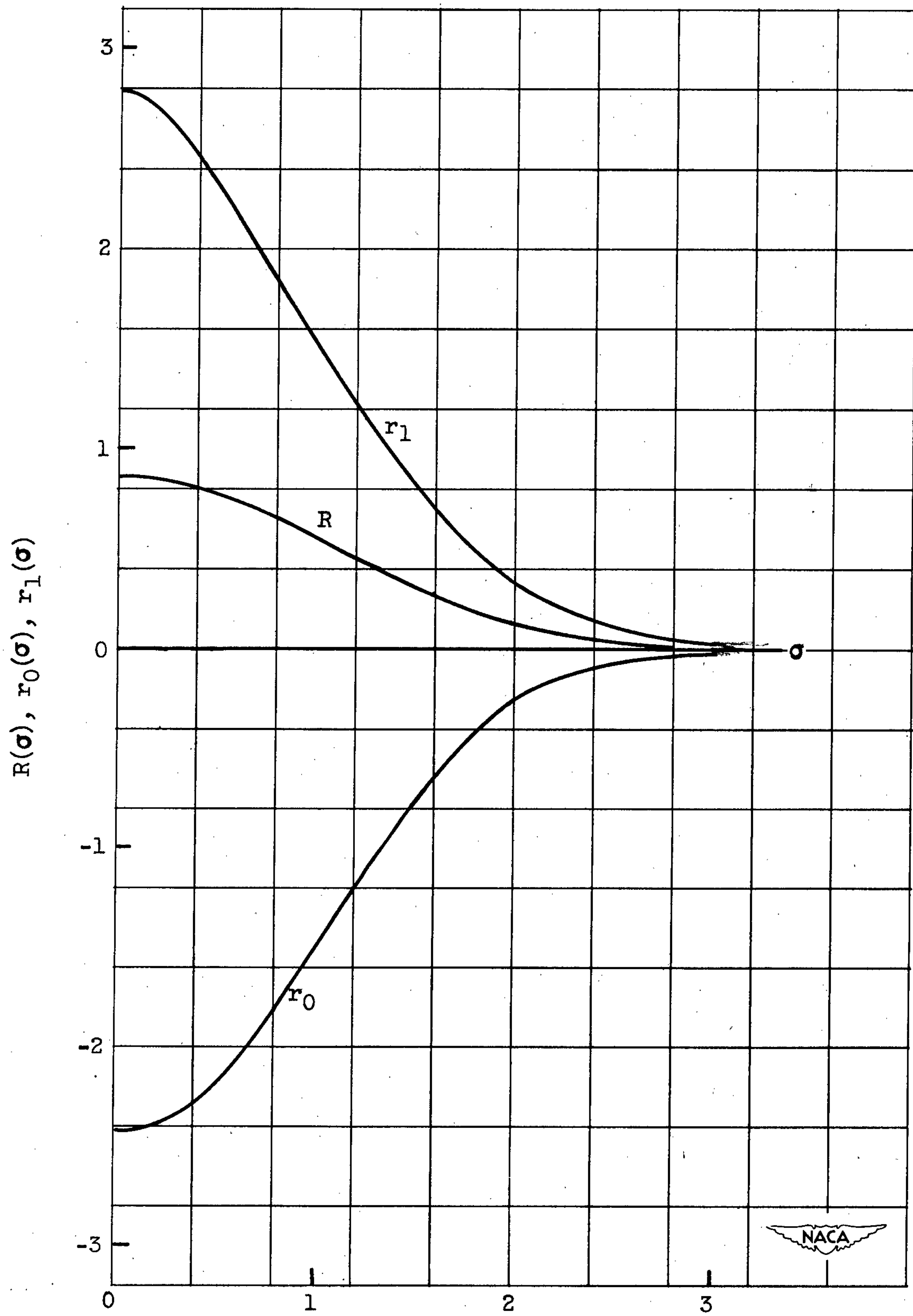

Figure 3. - Functions associated with temperature profile.